3 × 3 Matrix for unitary optical systems

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We introduce a 3×3 matrix for the study of unitary optical systems. This 3×3 matrix is a submatrix of the 4×4 Mueller matrix. The elements of this 3×3 matrix are real, and thus complex-number calculations can be avoided. The 3×3 matrix is useful for illustrating the polarization state of an optical system. One can also use it to derive the conditions for linear and circular polarization output for a general optical system. New characterization methods for unitary optical systems are introduced. It is shown that the trajectory of the Stokes vector on a Poincaré sphere is either a circle or an ellipse as the optical system or input polarizer is rotated. One can use this characteristic circle or ellipse to measure the equivalent optical retardation and rotation of any lossless optical system.

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1. INTRODUCTION

Unitary optical systems are systems that are lossless and do not alter the intensity of the input light; they include polarization rotators, retardation wave plates, and twisted nematic liquid-crystal layers. The change in polarization through a unitary optical system is conventionally handled by 2×2 Jones calculus.[1] The elements of the Jones matrix are in general complex. The Jones vector is a complex 2-vector that describes the polarization state of the electric field. In general, three numbers (the complex amplitude of the electric fields in the x and in the y directions and their relative phase) are needed for definition of a polarization state for a completely polarized light beam.

Any lossless linear optical system can be represented by a unitary Jones matrix. It has been pointed out that all unitary matrices can be represented as a combination of a polarization rotator and a retardation plate.[2] Simon and Makunda[3] and Bagini *et al.*[4] invented several devices ("gadgets") for modeling polarization optics of unitary systems. Most of these studies made use of the 2×2 Jones matrix formulation or the SU(2) transformations. In particular, the Jones matrix of a liquid-crystal cell has been studied extensively.[5] Although this Jones calculus has been applied successfully to many optical systems, it suffers from the drawbacks that it is not directly amenable to physical interpretation vis-à-vis the Poincaré sphere and that calculations that involve complex numbers are needed.

The 2×2 Jones matrix and Jones vector can be extended to a 4×4 Mueller matrix and a 4-Stokes vector.[6,7] A Stokes vector is a generalized description of the polarization state of light that allows for absorption as well, so the vector need not have unity length. Partial polarization can also be accommodated. A Stokes vector can be visualized as a vector in the Poincaré sphere and is quite a useful concept in physical interpretations. In this paper we introduce the 3×3 matrix calculus for polarization-state calculations for the special case of unitary optical systems. It is suitable for optical systems for which there is no optical absorption and the input optical element is always a linear polarizer, so that the light is always totally polarized. This 3×3 matrix can be considered a subset of the 4×4 Mueller matrix confined to unitary systems, and its use is more convenient. Similarly to the 4×4 case, the 3-Stokes vector and this 3×3 matrix also have real matrix elements. Moreover, the new Stokes vectors that describe the polarization states of light can also be clearly represented on the Poincaré sphere, and, in particular, as projections onto the S_1–S_2 plane. They are homomorphic to the SO(3) group.

Similarly to 2×2 and 4×4 calculus, the final output intensity of light in the 3×3 calculus is obtained by examining the final Stokes vector. In the following discussions, formulation of a 3×3 matrix for both transmission and reflection of unitary optical systems will be discussed. We demonstrate the usefulness of this 3×3 matrix for polarization-state manipulation in addition to intensity calculations by deriving some important polarization-state conversion formulas. Specific conditions for linearly and circularly polarized output light are derived.

We also introduce the interesting concept of the characteristic circle and the characteristic ellipse for any unitary optical system. Application of the new matrix methods to characterization of optical systems is also introduced. These new characterization methods are applicable to all unitary optical systems and in particular to twisted nematic liquid-crystal cells.

2. FORMULATION OF THE 3 × 3 MATRIX

It was shown previously[6,7] that, for every unitary Jones matrix $\mathbf{M}_J$, there is a 4×4 Mueller matrix counterpart $\mathbf{M}_M$ that can be obtained through the following transformation:

$$\mathbf{M}_M = \mathbf{T} \cdot (\mathbf{M}_J \otimes \mathbf{M}_J{}^*) \cdot \mathbf{T}^{-1}, \tag{1}$$

where

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$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix}, \quad (2)$$

and the direct product of two 2×2 Jones matrices is given by

$$\mathbf{M}_J \otimes \mathbf{M}_J{}^* \begin{bmatrix} m_{11}m_{11}^* & m_{11}m_{12}^* & m_{12}m_{11}^* & m_{12}m_{12}^* \\ m_{11}m_{21}^* & m_{11}m_{22}^* & m_{12}m_{21}^* & m_{12}m_{22}^* \\ m_{21}m_{11}^* & m_{21}m_{12}^* & m_{22}m_{11}^* & m_{22}m_{12}^* \\ m_{21}m_{21}^* & m_{21}m_{22}^* & m_{22}m_{21}^* & m_{22}m_{22}^* \end{bmatrix}, \quad (3)$$

where m_{ij} are the matrix elements of Jones matrix $\mathbf{M}_J$.

It can be shown that the most general unitary 2×2 matrix can be written in the form

$$\mathbf{M}_J = \begin{bmatrix} a + ib & -c + id \\ c + id & a - ib \end{bmatrix}, \quad (4)$$

where a, b, c, and d are real numbers and $a^2 + b^2 + c^2 + d^2 = 1$. Thus there are three independent variables for the Jones matrix of a unitary optical system.

Physically, the most general unitary Jones matrix is that of a combination of a polarization rotator and a retardation plate, which is also the Jones matrix of a liquid-crystal (LC) cell. In this paper we are interested in applying this 3×3 matrix to LC cells. Therefore, in the following, we can discuss the general unitary matrix in the language of a LC cell for which there is a twist angle ϕ and the retardation value of the LC cell is given by $\delta = \pi d \Delta n/\lambda$, where d is the LC cell's thickness and Δn is the value of the LC's birefringence. For a general twisted nematic LC cell whose input director is parallel to the x axis, the Jones matrix elements are given by[8,9]

$$a = \cos\beta\cos\phi + \frac{\phi}{\beta}\sin\beta\sin\phi,$$

$$b = -\frac{\delta}{\beta}\sin\beta\cos\phi,$$

$$c = \cos\beta\sin\phi - \frac{\phi}{\beta}\sin\beta\cos\phi,$$

$$d = -\frac{\delta}{\beta}\sin\beta\sin\phi, \quad (5)$$

where $\beta^2 = \phi^2 + \delta^2$. Notice that this is just a special case of the general matrix $\mathbf{M}_J$. There are only two independent variables in Eqs. (5). In general, the input director of the LC cell can make an angle α with the x axis. This provides the third independent variable.

Going back to the general a, b, c, d representation, after some algebra, Mueller matrix $\mathbf{M}_M$ for a general unitary optical system can be found to be

$$\mathbf{M}_M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - 2(c^2 + d^2) & 2(bd - ac) & -2(ad + bc) \\ 0 & 2(ac + bd) & 1 - 2(b^2 + c^2) & 2(ab - cd) \\ 0 & 2(ad - bc) & -2(ab + cd) & 1 - 2(b^2 + d^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & A & B & C \\ 0 & D & E & F \\ 0 & G & H & K \end{bmatrix}. \quad (6)$$

The 4-Stokes vector has the standard form

$$\mathbf{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}, \quad (7)$$

where S_0 always equals 1 for completely polarized light. Now, for the unitary optical system, the first column of the Mueller matrix is always $(1 \quad 0 \quad 0 \quad 0)^{\mathrm{T}}$, and the first row is always $(1 \quad 0 \quad 0 \quad 0)$. Moreover, $\mathbf{S}_0$ in the Stokes vector is always unity. Thus we can ignore the first row and the first column and simplify matrix $\mathbf{M}_M$ as

$$\mathbf{M}_M = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & K \end{bmatrix}. \quad (8)$$

We have to make the corresponding simplification of the Stokes vector by ignoring the first element of the 4-Stokes vector. Thus the new Stokes vector is

$$\mathbf{S} = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}. \quad (9)$$

This is the basis of the new 3×3 matrix calculus. It is less tedious to treat this 3×3 calculus than the 4×4 calculus. As in the case of the 4-Stokes vector, the new 3-Stokes vector comprises simply Cartesian coordinates of a unity radius Poincaré sphere. Thus light polarized linearly at α to the x axis is represented by $(\cos 2\alpha \quad \sin 2\alpha \quad 0)^{\mathrm{T}}$, and circularly polarized light is represented by $(0 \quad 0 \quad \pm 1)^{\mathrm{T}}$.

Note that all elements of the 3×3 matrix and the 3×1 Stokes vector are real. The forms of some important 3×3 matrices are given here. The 3×3 matrix of a wave plate with a retardation of 2Γ and the slow axis along the x axis is

$$WP(\Gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\Gamma & -\sin 2\Gamma \\ 0 & \sin 2\Gamma & \cos 2\Gamma \end{bmatrix}. \quad (10)$$

For a polarization rotator that rotates the polarization of an incoming wave by an angle θ, the 3×3 matrix is

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (11)$$

We observed that the 3 × 3 matrix of a wave plate and that of a rotator are, respectively, a three-dimensional rotation matrix along the S_1 and S_3 axes. In fact, this should not be surprising if one recognizes that the unitary group SU(2) and the rotation group SO(3) are homomorphic.[10,11] One can equate the 2 × 2 Jones matrix to the SU(2) group and the 3 × 3 matrix to the SO(3) rotational group. Finally, it should be noted that the polarizer matrix is not unitary and that it does not have a 3 × 3 matrix representation.

3. CALCULUS OF THE 3 × 3 MATRIX

The 3-Stokes vectors are transformed by the 3 × 3 matrix in the usual manner:

$$\mathbf{S}_{\text{out}} = \mathbf{M}_M \mathbf{S}_{\text{in}}. \qquad (12)$$

Unlike for the Jones vector, the transmission and reflection of an optical system represented by Figs. 1 and 2 cannot be calculated from the magnitude of the Stokes vector. For a 4-Stokes vector, the magnitude is simply the first element of the vector. It can be shown, after some algebra, that in the 3 × 3 formulation the transmission of light for a unitary optical system between two polarizers at angles α and γ can be written as

$$\mathbf{T} = 0.5 + 0.5(\cos 2\gamma \quad \sin 2\gamma \quad 0) \cdot \mathbf{M} \cdot \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix}, \qquad (13)$$

where $\mathbf{M} = M_n M_{n-1} \dots M_2 M_1$. The M_i element here can be a wave plate, a polarization rotator, or an arbitrary twisted nematic LC cell. In Eq. (13), $(\cos 2\alpha \quad \sin 2\alpha \quad 0)^{\mathrm{T}}$ represents the input polarizer at angle α, and $(\cos 2\gamma \quad \sin 2\gamma \quad 0)$ represents the output polarizer at angle γ relative to the x axis.

The maximum transmission in Eq. (13) is unity when the final output polarization is linear at angle γ. Equation (13) can be derived by using the 4 × 4 Mueller matrix of a polarizer[9]:

$$\mathrm{Pol}(\gamma) = \frac{1}{2}\begin{bmatrix} 1 & \cos 2\gamma & \sin 2\gamma & 0 \\ \cos 2\gamma & \cos^2 2\gamma & \cos 2\gamma \sin 2\gamma & 0 \\ \sin 2\gamma & \cos 2\gamma \sin 2\gamma & \sin^2 2\gamma & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (14)$$

where the first element of the 4-Stokes vector represents the light intensity.

Similarly, it can be shown that, for a reflective unitary optical system as shown in Fig. 2, the reflectance is given by

$$R = 0.5 + 0.5(\cos 2\alpha \sin 2\alpha \ \ 0) \cdot \mathbf{M}^P \mathbf{M} \cdot \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix}, \qquad (15)$$

where $\mathbf{M}^P$ is given by

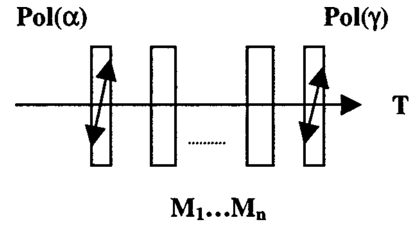

Fig. 1. Unitary optical system between two polarizers (Pols). All angles are relative to the x axis. M_i are arbitrary unitary optical elements.

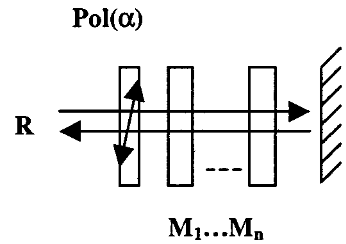

Fig. 2. Reflective optical system with unitary optical elements M_i and a single front polarizer.

$$\mathbf{M}^p = \begin{bmatrix} M_{11} & M_{21} & -M_{31} \\ M_{12} & M_{22} & -M_{32} \\ -M_{13} & -M_{23} & M_{33} \end{bmatrix}. \qquad (16)$$

Equation (15) is obtained when we consider that the Jones matrix of a unitary optical element for a reversed light path is the transpose of the forward light path.[13] In fact, $\mathbf{M}^P$ here is the Mueller matrix counterpart of the transpose of Jones matrix $\mathbf{M}_J$. It is important to note that the transpose of a Mueller matrix does not correspond to the transpose of its Jones matrix counterpart. Matrix $\mathbf{M}^P$ can also be written in the form[14,15]

$$\mathbf{M}^P = \mathbf{O}\mathbf{M}^T\mathbf{O}^{-1} \qquad (17)$$

where $\mathbf{M}^T$ is the transpose of $\mathbf{M}$ and $\mathbf{O}$ is

$$\mathbf{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \qquad (18)$$

To obtain Eq. (16) one obtains the transpose of $\mathbf{M}_J$ by just reversing the sign of c in Eq. (5). Therefore reversing the sign of c in Eq. (5) gives the $\mathbf{M}_J^T$ Mueller matrix counterpart $\mathbf{M}^P$.

The above equations represent calculations of optical transmission and reflection through a unitary optical system by use of the new 3 × 3 Mueller calculus. It should be emphasized that all the matrix and vector elements are real. The demand on computation time is correspondingly diminished.

4. POLARIZATION CONVERSION

The state of polarization for any polarized light can be illustrated easily in a Poincaré sphere representation.[12] The close relationship between the 3 × 3 calculus and the state of polarization is remarkable. In this section we examine the induced change of polarization state by a general unitary optical system, using the 3 × 3 calculus developed above. The most-wanted output states are the linear and circular polarization states. If we want to have a linear polarization output for a particular unitary optical system, then

$$\begin{pmatrix} \cos 2\gamma \\ \sin 2\gamma \\ 0 \end{pmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & K \end{bmatrix} \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix}. \tag{19}$$

Therefore the condition for linear output is

$$G \cos 2\alpha + H \sin 2\alpha = 0. \tag{20}$$

Equation (20) has two solutions:

$$G = H = 0, \tag{21}$$

$$\tan 2\alpha = -G/H. \tag{22}$$

Equations (21) and (22) are the first and the second conditions for linear output, termed the LP1 and the LP2 solutions, respectively. It is of interest to note that the LP1 solution is independent of input linear polarization angle α and exists only for special unitary optical systems. It may not exist for all systems. However, the LP2 solution always exists in the sense that one can always find α in terms of the optical system parameters. Therefore we have the following theorems:

Theorem 1: For any unitary optical system, if the input light is linearly polarized there always exists an orientation of the system such that the output is also linearly polarized.[6] The orientation of the system is given by Eq. (22).

Theorem 2: There are some unitary optical systems such that a linearly polarized input results in a linearly polarized output, independently of the orientation of the optical systems. The parameters for those systems are given by Eq. (21).

These theorems are valid for all unitary (lossless) optical systems. In particular, they are true for LC cells. For the LC cell the physical interpretation of the above theorems is quite interesting. They are discussed further in Section 5 in terms of equivalence to a retardation plate–polarization rotator combination.

If the input is linearly polarized and a circular output is wanted, we must have

$$\begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & K \end{bmatrix} \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix} \tag{23}$$

or

$$\begin{bmatrix} A & D & G \\ B & E & H \\ C & F & K \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} = \begin{pmatrix} \cos 2\alpha \\ \sin 2\alpha \\ 0 \end{pmatrix}. \tag{24}$$

We obtained Eq. (24) by using the property that the transpose of $\mathbf{M}_M$ is also the inverse of itself, which is true for any unitary matrix.[16] Thus the following criteria for a circular polarization output can be obtained:

$$\tan 2\alpha = H/G, \tag{25}$$

$$K = 0. \tag{26}$$

Equations (25) and (26) are the conditions for a circular polarization output that is termed the CP solution. Therefore, for CP output, K must be zero and that the input polarizer must be at angle α according to Eq. (25). Since K is a parameter related to the optical system only, Eq. (26) is a useful check on whether an optical system can produce a CP output. Thus we have theorem 3:

Theorem 3: There are some conditions on the parameters of a unitary optical system such that a linearly polarized input will result in a circularly polarized output. The conditions for the optical parameters are given by Eqs. (25) and (26).

Note that the LP2 solution always exists for any unitary system. LP1 and CP solutions require special conditions on the parameters of the system for these solutions to occur. Finally, we observe that, for deducing the linear polarization or CP output condition, only the third row of the resultant matrix $\mathbf{M}$, $(G\,H\,K)$, is needed. The results obtained here give useful insights into the output polarization of unitary optical systems. They can also help in the development of new optical systems for polarization manipulation, particularly for LC cells.

Table 1 is a summary of the polarization-conversion requirements given in both Mueller and Jones parameters. It is of interest to note that the equations for calculating input polarizer angle α for the LP2 and the CP modes are

Table 1. Conditions for Obtaining Linearly or Circularly Polarized Output for General Mueller and Jones Calculus

Output	Calculus	
	Mueller	Jones
Linear polarization		
LP1	$G = H = 0$	$a^2 + c^2 = 1,$ 0
LP2	$\tan 2\alpha_{\mathrm{LP2}} = -G/H$	$\tan 2\alpha_{\mathrm{LP2}} = \dfrac{ad - bc}{ab + cd}$
Circular polarization	K=0	$a^2 + c^2 = 1/2$
	$\tan 2\alpha_{\mathrm{CP}} = H/G$	$\tan 2\alpha_{\mathrm{LP2}} = \dfrac{ab + cd}{bc - ad}$

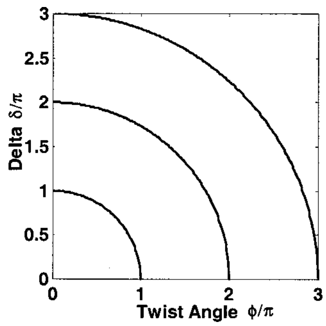

Fig. 3. Solution curves of Eq. (27) are circles in ϕ–δ space. Circles correspond to the order of solution; the smallest circle is for $N = 1$, and so on.

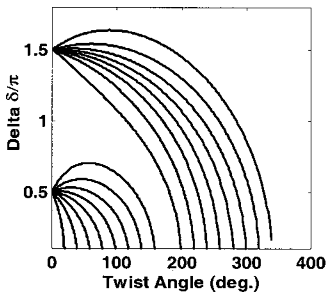

Fig. 4. LP2 solutions span the whole ϕ–δ space. The curves shown are the first two orders of solutions with different input polarizer angle α. The input polarizer angle begins at 10°–80° from left to right.

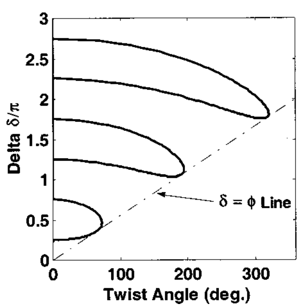

Fig. 5. CP solutions with Eq. (29) as α is varied. Curves are for different orders of the solution. No solution could be obtained for $\phi > \delta$.

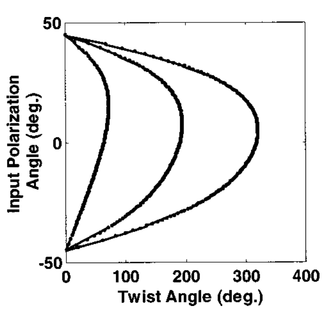

Fig. 6. For the CP solutions, one can plot the input polarizer angle α as a function of the twist angle. The three curves here represent the three curves in Fig. 5.

quite similar. This fact suggests that if we have a CP output obtained by solving Eq. (25), then, by rotating the linear input angle (or equivalently the unitary optical system) by an angle of ±45°, we get a linear polarization output of the LP2 type. But, of course, for the CP mode, an additional condition for the relation between a and c (or K) is required.

Specializing to the case of a LC cell, using the specific formulas for a, b, c, and d in Eq. (5), we can derive the polarization-conversion equations to be

for LP1

$$\sin\beta = 0, \tag{27}$$

for LP2

$$\tan 2\alpha_{\mathrm{LP2}} = \phi/\beta \tan\beta, \tag{28}$$

for CP

$$\frac{\delta}{\beta}\sin\beta = \frac{1}{\sqrt{2}}, \tag{29}$$

$$\tan 2\alpha_{\mathrm{CP}} = -\beta/\phi \cot\beta. \tag{30}$$

The output linear polarization angles γ for the LP1 and LP2 solutions are determined easily to be $\phi + \alpha$ and $\phi - \alpha$, respectively.

Note that Eqs. (5) were obtained assuming that the input director of the LC cell was along the x axis. The polarizer angle is α. If we regard the polarizer as fixed, then α can be treated as the rotation angle of the LC cell. Thus (α, ϕ, δ) are the three required parameters for the LC cell. (The 3 × 3 matrix needs three independent variables, in general.)

Equations (27)–(30) are useful for both LC cell design and measurement. The solution curves of the polarization-conversion equations for the general twisted nematic LC cell are shown in Figs. 3–6. Details of the application of these solutions are found in our previous publications.[17–19] Basically, they give the conditions in the LC cell for the LP1, LP2, and CP solutions to exist. For the LP1 and LP2 solutions, since they require only one condition on the optical parameters, the solution spaces are therefore surfaces in the (α, ϕ, δ) parameter space. For the CP solution, since there are two conditions imposed on the optical system the solution space is therefore a curve in the same (α, ϕ, δ) parameter space. Figures 3–6 can be treated as representations of the three-dimensional parameter space on a two-dimensional diagram. But LP2 solutions are unique in the sense that they always exist for any optical system, by a rotation of the entire optical system (Theorem one). Finally, we note that the results in this section are similar to the state-of-polarization analysis of Zhuang *et al.*, who used the 4 × 4 matrix.[20] However, the analysis of the polarization conversion in terms of the LP1, LP2, and CP solutions is much more elegant.

5. CHARACTERIZATION OF A UNITARY OPTICAL SYSTEM: EQUIVALENCE THEOREM

There are several equivalent theorems for unitary optical systems. Since the most general unitary matrix contains three independent parameters, any combination of optical elements that allows three independent parameters can potentially be an optical equivalent. In particular, it has been shown that any unitary optical system can be replaced by the combination of a retardation wave plate and a polarization rotator.[2] It has also been proved that it is possible to represent an arbitrary unitary optical system by two quarter-wave plates and a half-wave plate (the Simon–Mukunda gadget).[3,4] Another potential equivalence is that of a combination of a retardation plate and a half-wave plate or a quarter-wave plate. Here we discuss in detail the equivalence theorem with the polarization rotator and the wave plate, since it is more closely related to a LC layer.

For any unitary matrix $\mathbf{M}$,

$$\mathbf{M} = R(\chi)WP(\Gamma, \psi), \tag{31}$$

where χ is the rotation angle induced by the rotator and Γ and ψ are the phase retardation and the orientation, respectively, of the c axis of the equivalent wave plate. The order of the matrices in Eq. (31) is not important as long as the order is consistent. If the order is reversed, the values of the rotation angle and the direction of the c axis will be changed as well. As expected, there are three independent variables for a unitary optical system in Eq. (31). Of the three parameters, χ and Γ are independent of the reference coordinate system. χ is called the characteristic angle, and Γ is the characteristic phase of the unitary optical system.[8] In this section we introduce a new Stokes parameter method for the measurement of these two characteristic parameters, using the new 3×3 matrix introduced above.

Before we discuss the experimental techniques for determining Γ and χ, let us use this equivalence theorem to reexamine the LP conditions derived in Section 3. Physically, the LP1 condition corresponds to the case in which the equivalent retarder has a retardation of $2N\pi$ for integer values of N. In this case the optical system behaves as a pure polarization rotator. This is known as the integral wave-plate condition. The output polarization direction is simply given by $\gamma = \phi + \alpha$. For the LP2 condition the rotation of the optical system is such that the input polarizer is parallel to the c axis of the equivalent wave plate. Thus the output is also linearly polarized, with a direction $\gamma = \phi - \alpha$ (for a twisted nematic LC cell).[19] Note again that the LP2 solution is always present, whereas the LP1 solution requires special conditions on the optical system. If the optical system is a LC cell, the LP1 solution is simply the waveguiding or Gooch–Tarry mode of the LCD.[21,22]

To measure the values of Γ and χ, consider the experimental setup shown in Fig. 7. The input linearly polarized light is kept in the horizontal direction by a horizontal polarizer. Without loss of generality, we can assume that the c axis of the equivalent wave plate is along the x axis at the beginning of the experiment. If we rotate the unitary optical system by an angle θ, the output Stokes vector will be represented by

$$\mathbf{S}' = R(\chi) \cdot R(\theta) \cdot WP(\Gamma) \cdot R(-\theta) \cdot \mathbf{S}. \tag{32}$$

After some algebra, we get

$$\mathbf{S}' = R(\chi) \cdot \begin{pmatrix} \cos^2 2\theta + \sin^2 2\theta \cos 2\Gamma \\ \sin 2\theta \cos 2\theta (1 - \cos 2\Gamma) \\ \sin 2\Gamma \sin 2\theta \end{pmatrix} = R(\chi) \cdot \mathbf{S}''. \tag{33}$$

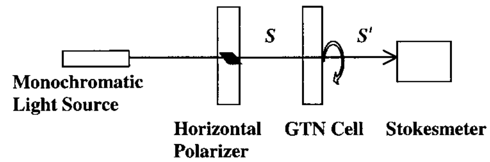

Fig. 7. Experimental setup for the Stokes parameter method of cell parameter measurement. $\mathbf{S}$ and $\mathbf{S}'$ are the Stokes vectors before and after the Gooch–Tarry nematic LC cell.

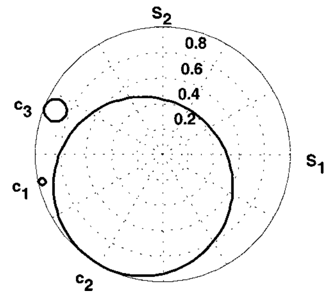

Fig. 8. Examples of characteristic circles on the $\mathbf{S}_1$–$\mathbf{S}_2$ plane: $c1$, first minimum twisted nematic cell ($\phi = 90°$, $d\Delta n = 0.5\,\mu$m); $c2$, second minimum twisted nematic cell ($\phi = 90°$, $d\Delta n = 1.0\,\mu$m); $c3$, standard twisted nematic LC cell ($\phi = 240°$, $d\Delta n = 0.85\,\mu$m). Wavelength, 632.8 nm.

Eliminating θ from $\mathbf{S}_1''$ and $\mathbf{S}_2''$ of Eq. (33) gives

$$(\mathbf{S}_1'' - \cos^2 \Gamma)^2 + \mathbf{S}_2''^2 = \sin^4 \Gamma. \tag{34}$$

Equation (34) is recognized to be the parametric equation of a circle on the $\mathbf{S}_1$–$\mathbf{S}_2$ plane with radius $\sin^2 \Gamma$ and centered at $(\cos^2 \Gamma, 0)$. This circle touches the Poincaré sphere at position (1, 0), i.e., along the $\mathbf{S}_1$ axis. Now the trajectory of $\mathbf{S}$ on the $\mathbf{S}_1$–$\mathbf{S}_2$ plane is simply the projection of the Stokes vector onto the $\mathbf{S}_1$–$\mathbf{S}_2$ plane, so the projection of $\mathbf{S}'$ on the $\mathbf{S}_1$–$\mathbf{S}_2$ plane can be represented by the same circle but rotated about the origin by an angle χ. We call this the characteristic circle of the optical system. If one measures the Stokes parameters of the output light in Fig. 7 by using the Stokesmeter as the optical system is rotated, the result will be the characteristic circle. Figure 8 shows examples of this characteristic circle for some optical systems (LC cells). Note that the radius of $c1$ for the twisted nematic LC display is very small. This makes sense since it is near the waveguiding Gooch–Tarry mode of the LC display. Thus it is nearly a perfect polarization rotator, with very small birefringence. The $c2$ mode has a large radius because of the size of the wavelength assumed.

From Eq. (34) it can be seen that the radius of the characteristic circle is determined by phase Γ of the equivalent wave plate. The center of this characteristic circle is offset from the x axis by an angle χ. Therefore, by measuring the azimuth and the radius of the characteristic circle, one can determine the characteristic angle χ and phase Γ; the unitary optical system is thus defined uniquely.

In the above analysis, the unitary optical system is required to rotate during the measurement. However sometimes it may be impossible for some system under test to be rotated. In that case, the input polarizer should be rotated instead. It is interesting to note that in this case, the resulting Stokes vector draws an ellipse rather than a circle on the $\mathbf{S}_1$–$\mathbf{S}_2$ plane.

Then the $\mathbf{S}'$ and $\mathbf{S}''$ matrices are given by

$$\mathbf{S}' = R(\chi) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\Gamma & -\sin 2\theta \\ 0 & \sin 2\Gamma & \cos 2\Gamma \end{bmatrix} \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \\ 0 \end{pmatrix} = R(\chi) \cdot \mathbf{S}'', \tag{35}$$

where θ represents the direction of the input polarizer and the wave plate's c axis is along the x axis. Thus we have

$$S_1'' = \cos 2\theta,$$

$$S_2'' = \cos 2\Gamma \sin 2\theta. \tag{36}$$

Eliminating θ from S_1'' and S_2'' in Eqs. (36) gives

$$S_I''^2 + \left(\frac{S_2''}{\cos 2\Gamma}\right)^2 = 1, \tag{37}$$

which is the equation for an ellipse on the $\mathbf{S}_1$–$\mathbf{S}_2$ plane with ellipticity $e = \cos 2\Gamma$. The center of the ellipse is at (0, 0), and the major axis of this ellipse touches the Poincaré sphere at (1, 0). Similarly as for the characteristic circle, the measured characteristic ellipse, given by $\mathbf{S}'$, is obtained from $\mathbf{S}''$ by a rotation of the axes by an angle χ. Therefore, by measuring ellipticity e and offset angle χ of the characteristic ellipse, we can define the unitary optical system completely. Several examples of this characteristic ellipse are shown in Fig. 9. We can obtain the characteristic angle χ here by measuring the direction of linear input polarization when the ellipse touches the $\mathbf{S}_1$–$\mathbf{S}_2$ circle. It is also interesting to note that ellipticity e here can be proved to be identical to parameter K in Eq. (8).

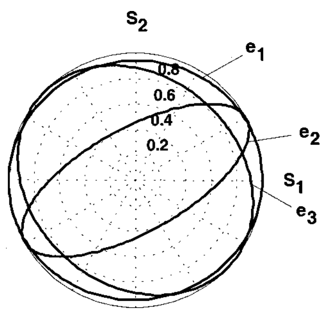

Fig. 9. Characteristic ellipses for the same LC cells as in Fig. 8.

If the unitary optical system is a twisted nematic LC cell, then twist angle ϕ and retardation $d\Delta n$ can be determined by solution of the equations that relate ϕ, $d\Delta n$ and Γ, χ. A summary of the results is given in Table 2. The relationship between the unitary optical system parameters (a, b, c, d) and the characteristic parameters (χ, Γ) can be obtained, after some algebra, from Eq. (31). We intend to publish details of the derivation of the equations in Table 2 and also some experimental results subsequently.

6. CONCLUSIONS

In this paper we have given the formulation of a 3 × 3 matrix for calculation of the polarization state of light as it traverses any number of unitary optical elements. This formulation has the advantage that all the matrix elements are real numbers. This point is quite important in optical modeling calculations, as manipulation of complex numbers will inevitably be more time consuming. Both transmission between arbitrary polarizer directions and reflection from a rear-mirror system have been considered in this paper.

We have shown that this 3 × 3 matrix is particularly useful in deriving the conditions under which a linear polarization input results in a linear or circular polarization output. Three useful conditions were presented, including the LP1, LP2, and CP modes. The LP2 condition exists for all unitary optical systems. That is, for any lossless optical system, a linearly polarized input will always produce a linearly polarized output, provided that the optical system is oriented properly. This is true for any unitary optical system, including a voltage biased or unbiased twisted nematic liquid-crystal cell. This is an interesting observation. It states that a general twisted nematic LC cell can also produce a perfect linearly polar-

Table 2. Relationship of Characteristic Parameters of the Unitary Optical System to Various Matrix Elements and to LC Cell Parameters

Characteristic Parameter	Relationship to the Unitary Optical System	Relationship to the LC Cell
Angle χ	$\tan\chi = c/a$	$\tan\chi = \dfrac{\beta\tan\phi - \phi\tan\beta}{\beta + \phi\tan\phi\tan\beta}$
Phase Γ	$\cos^2\Gamma = \dfrac{K+1}{2}$	$\sin^2\Gamma = \dfrac{\delta^2}{\beta^2}\sin^2\beta$
	or $\cos^2\Gamma = (a^2 + c^2)$	
Radius R	$R = (1 - K)/2$	$R = \dfrac{\delta^2}{\beta^2}\sin^2\beta$
	or $R = 1 - (a^2 + c^2)$	
Ellipticity e	$e = K$	$e = 1 - \dfrac{2\delta^2}{\beta^2}\sin^2\beta$
	or $e = 2(a^2 + c^2) - 1$	

ized output, under any applied voltage. This fact is important in the modeling and design of LC displays.

We also used the new 3×3 matrix to derive characterization methods of any unitary optical system. It was shown that the Stokes vector will describe either a circle or an ellipse on the equatorial plane of the Poincaré sphere as the optical system or the direction of input polarization is rotated. The radius as well as the position of the trajectory will give the retardation and the rotational angle of the equivalent retarder and polarization rotator of the optical system. All these results apply to any unitary optical component, including twisted nematic LC layers. This equivalence method is powerful for use in determining the optical properties of any lossless optical system.

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